The Generalized Multiplicative Gradient Method for A Class of Convex Symmetric Cone Optimization Problems

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Motivating Example: Positron Emission Tomography

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- Multiplicative gradient method: $x^0 \in \mathsf{ri}\,\Delta_n$

$$x^{t+1} = x^t \circ \nabla F(x^t) \quad \equiv \quad x_i^{t+1} := x_i^t \nabla_i F(x^t), \quad \forall\, i \in [n].$$

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- (MG) does not fall under any "well-known" optimization frameworks, e.g., Newton-type method, mirror descent, etc.

The Mystery of MG

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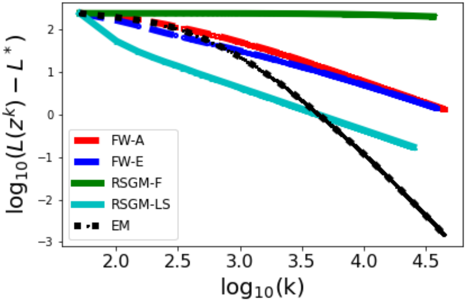
- Impressive numerical performance: $x^0 = (1/n)e$



FW-A & FW-E [Dvu20; ZF20]: Frank-Wolfe (FW) method for logarithmically-homogeneous self-concordant barriers (with adaptive stepsize and exact line search)

RSGM-F & RSGM-LS [BBT17; LFN18]: Relatively smooth gradient method (with fixed stepsize and backtracking line search)

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  - And what is the interaction between the complexity of (MG) and the problem structure?

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- D-optimal design
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  - Semidefinite relaxation of Boolean QP
- ▷ In all of these applications, the objective functions involve "$\ln(\cdot)$", and hence are neither Lipschitz nor smooth (i.e., have Lipschitz gradients) on the feasible sets.
- ▷ Certain first-order methods for these applications have been developed recently [Nes11; BBT17; LFN18; Dvu20; ZF20] — our generalized MG method contributes to this line of research from a different viewpoint.

D-Optimal Design (D-OPT)

$$\max_x F(x) := \ln\det\left(\sum_{i=1}^n x_i a_i a_i^\top\right) \quad \text{s.t.} \quad x \in \Delta_n \qquad \text{(D-OPT)}$$

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- ▷ MG method: $\boxed{x^0 \in \mathsf{ri}\,\Delta_n, \quad x^{t+1} = x^t \circ (m^{-1}\nabla F(x^t))}$

Quantum State Tomography (QST)

$$\begin{aligned} &\max_X \; F(X) := m^{-1} \textstyle\sum_{j=1}^q n_j \ln(\langle X, a_j a_j^H \rangle) \\ &\text{s.t.} \quad X \in \mathbb{H}_+^n, \;\; \mathrm{tr}(X) = 1 \end{aligned} \tag{QST}$$

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(For any $X = \sum_{i=1}^{n} \lambda_i u_i u_i^H \succ 0$, $\ln(X) := \ln(\lambda_i) u_i u_i^H$.)

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$$\begin{aligned} \max_X \quad & F(X) := \ln\left(\textstyle\sum_{i=1}^n \langle X, r_i r_i^\top\rangle^{1/2}\right)^2 \\ \text{s. t.} \quad & X \in \mathbb{S}_+^n,\ \mathrm{tr}(X) = 1 \end{aligned} \tag{RBQP}$$

  where $A = R^\top R$ and $R := [r_1 \ \cdots \ r_n]$, and $\mathbb{S}_+^n$ denotes the cone of $n \times n$ real symmetric PSD matrices.

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Comparison of Computational Guarantees

RSGM [BBT17; LFN18]: Relatively smooth gradient method
FW [Dvu20; ZF20]: FW method for logarithmically-homogeneous self-concordant barriers
MG: (Generalized) Multiplicative gradient method (this work)
BSG [Nes11]: Barrier subgradient method

Table 1: Comparison of operations complexities (with $x^0 = (1/n)e$ or $X^0 = (1/n)I_n$)

	RSGM	FW	MG	BSG	Regime
PET	$O\left(\frac{mn^2}{\varepsilon}\ln\left(\frac{\ln(n)}{\varepsilon}\right)\right)$	$O\left(\frac{m^2 n}{\varepsilon}\right)$	$O\left(\frac{mn\ln(n)}{\varepsilon}\right)$	$O\left(\frac{mn^2}{\varepsilon^2}\ln^2\left(\frac{n}{\varepsilon}\right)\right)$	$n = O(\exp(m))$
D-OPT	$O\left(\frac{mn^2}{\varepsilon}\ln\left(\frac{\ln(n/m)}{\varepsilon}\right)\right)$	$O\left(\frac{m^2 n}{\varepsilon}\right)$	$O\left(\frac{m^2 n\ln(n)}{\varepsilon}\right)^\dagger$	$O\left(\frac{m^2 n^2}{\varepsilon^2}\ln^2\left(\frac{n}{\varepsilon}\right)\right)$	
QST	x?	$O\left(\frac{m^2 n^2}{\varepsilon}\right)$	$O\left(\frac{mn^2\ln(n)}{\varepsilon}\right)^\ddagger$	$O\left(\frac{mn^3}{\varepsilon^2}\ln^2\left(\frac{n}{\varepsilon}\right)\right)$	$n = O(\exp(m))$
RBQP	x?	x?	$O\left(\frac{n^3\ln(n)}{\varepsilon}\right)$	$O\left(\frac{n^4}{\varepsilon^2}\ln^2\left(\frac{n}{\varepsilon}\right)\right)$	

† [Coh19] ‡ [LCL21]

A General Problem Class

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\begin{aligned}
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- ▷ Let Γ be a regular cone in a (finite-dimensional) real vector space $\mathbb{Y}$, and $f : \Gamma \to \mathbb{R} \cup \{-\infty\}$ is a closed and strictly concave function:
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- f is C^3 and θ-logarithmically-homogeneous (θ-LH) on $\mathsf{int}\,\Gamma$:

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▷ $\mathcal{K} \subseteq \mathbb{V}$ is a symmetric cone, where $\mathbb{V}$ is a real inner-product space:

- self-dual: $\mathcal{K} = \mathcal{K}^*$
- homogeneous: for all $x, y \in \mathsf{int}\,\mathcal{K}$, there exists a (linear) automorphism T on $\mathcal{K}$ such that $\mathsf{T}x = y$

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- For $x \in \mathbb{V}$, $\operatorname{tr}(x)$ denotes the "trace" of x:
  - $\mathbb{V} = \mathbb{R}^n$: $\mathcal{K} = \mathbb{R}^n_+$, $\operatorname{tr}(x) := \sum_{i=1}^n x_i$
  - $\mathbb{V} = \mathbb{S}^n$: $\mathcal{K} = \mathbb{S}^n_+$, $\operatorname{tr}(X) := \sum_{i=1}^n \lambda_i(X)$

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- For $x \in \mathbb{V}$, $\operatorname{tr}(x)$ denotes the "trace" of x:
  - $\mathbb{V} = \mathbb{R}^n$: $\mathcal{K} = \mathbb{R}^n_+$, $\operatorname{tr}(x) := \sum_{i=1}^n x_i$
  - $\mathbb{V} = \mathbb{S}^n$: $\mathcal{K} = \mathbb{S}^n_+$, $\operatorname{tr}(X) := \sum_{i=1}^n \lambda_i(X)$
- $\mathsf{A} : \mathbb{V} \to \mathbb{Y}$ is a linear operator such that

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- ▷ Under these conditions, we know that $\nabla F : \mathsf{int}\,\mathcal{K} \to \mathsf{int}\,\mathcal{K}$.
- ▷ Finally, we require F to be *log-gradient convex* (more details later).

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▷ We call $(\mathbb{V}, \circ)$ *Euclidean* if $\mathrm{tr}(x^2) > 0$ for $x \neq 0$, and we can define an inner product $\langle x, y \rangle := \mathrm{tr}(x \circ y)$ that is *associative*:

$$\langle x \circ y, z \rangle := \langle x, y \circ z \rangle, \quad \forall\, x, y, z \in \mathbb{V}.$$

Spectral Decomposition

▷ Any x in a EJA $\mathbb{V}$ has a spectral decomposition $x = \sum_{i=1}^{n} \lambda_i(x) q_i(x)$:

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Connection between symmetric cones and EJA:

For each symmetric cone $\mathcal{K}$, there exists a unique EJA $\mathbb{V}$ such that $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ and

▷ $x \in \mathcal{K} \Longleftrightarrow \lambda_1(x), \ldots, \lambda_n(x) \geq 0$

▷ $x \in \mathsf{int}\, \mathcal{K} \Longleftrightarrow \lambda_1(x), \ldots, \lambda_n(x) > 0$

Generalized MG Method

$$\boxed{\begin{aligned} \text{Input}: &\quad x^0 \in \mathsf{ri}\,\mathcal{C} \;\; (:= \mathsf{int}\,\mathcal{K} \cap \{x : \mathrm{tr}(x) = 1\}) \\ \text{Iterate}: &\quad \hat{x}^{t+1} := \exp\{\ln(x^t) + \ln(\theta^{-1}\nabla F(x^t))\}, \\ &\quad x^{t+1} := \hat{x}^{t+1}/\mathrm{tr}(\hat{x}^{t+1}). \\ \text{Output}: &\quad \bar{x}^T := (1/T)\textstyle\sum_{t=0}^{T-1} x^t \end{aligned}} \tag{GMG}$$

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▷ In general, (GMG) updates both eigenvalues and the Jordan frame, and specializes to all the methods we've seen earlier.

Convergence Rate of (GMG)

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- ▷ The optimal choice for the above bound is $x^0 = (1/n)e$, and we have

$$F^* - F(\bar{x}^T) \le \frac{\theta \ln(n)}{T}, \qquad \forall\, T \ge 1$$

(Recall that n is the rank of the EJA associated with $\mathcal{K}$.)

The Class of Gradient Log-Convex Functions

Let $\mathrm{PI}(\mathbb{V}) := \{x \in \mathbb{V} : x^2 = x, \mathrm{tr}(x) = 1\}$ be the primitive idempotents in $\mathbb{V}$.

We call $F : \mathcal{K} \to \mathbb{R} \cup \{-\infty\}$ in (P) *log-gradient convex* if

$$\forall u \in \mathrm{PI}(\mathbb{V}), \quad x \mapsto \langle \ln \nabla F(x), u \rangle \quad \text{is convex on } \mathsf{int}\,\mathcal{K} \tag{GLC}$$

Recall that $F = f \circ \mathsf{A}$, some examples of f that satisfy (GLC):

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- $\mathcal{K}$ is any *representable* symmetric cone and $\Gamma = \mathbb{R}^m_+$, then
  - $f(y) = \sum_{j=1}^m w_j \ln y_j$, for all $y > 0$ and $w \in \mathsf{ri}\,\Delta_m$ (includes QST).
  - $f(y) = \ln \left\lvert\!\left\lvert\!\left\lvert y \right\rvert\!\right\rvert\!\right\rvert_p$, for all $y > 0$ and $p \in (0, 1]$ (includes RBQP).

Thank you!

Proof Sketch

Lemma 1 (An important conic inequality)

For all $t \geq 0$*, we have* $\mathrm{tr}(\hat{x}^{t+1}) \leq 1$ *and hence*

$$\ln(x^{t+1}) \succeq_{\mathcal{K}_1} \ln(x^t) + \ln(\nabla F(x^t)).$$

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To telescope, need the assumption that $\ln \nabla F(\cdot)$ is convex on $\mathsf{ri}\, \mathcal{C}$ (w.r.t. $\mathcal{K}_1$).

Lemma 2 (A growth bound of F)

For any $x \in \mathsf{ri}\, \mathcal{C}$ *and any optimal solution* x^*, $F^* - F(x) \leq \ln(\langle \nabla F(x), x^* \rangle)$.

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